MATH 144

Calculus for the Physical Sciences I

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These are notes for MATH 144 offered at the University of Alberta.

Preface

These notes are not meant to be complete lectures notes for MATH 144. Rather, they are structured to accompany the lectures offered at the University of Alberta.

Each lecture in the course corresponds to a subsection in the notes. Further, each subsection is structured as follows:

- A brief summary of the content of the section;
- The learning objectives of the section;
- The instructional videos, which should be watched before the live lecture;
- A summary section of the key concepts covered in the section;
- A reference to further readings in the CLP calculus textbooks.

The idea is that you should watch the videos, review the key concepts, and complete the pre-class quiz on eClass before the associated lecture. The live lecture will then be devoted mostly to examples and problem solving. The resulting in-class notes, which will accompany this document by providing many solved examples, will be posted on eClass.

Let us now delve into the beauty and depth of calculus!

Contents

Chapter 1

Review

In this section we review things that every calculus student should know (but is afraid to ask).

Objectives

You should be able to:

- Perform algebraic manipulations such as factorization, simplification, rationalization and completion of the square.
- Determine and state the domain and range of basic functions (root, rational) using interval notation.
- Determine whether a curve in the plane represents the graph of a function using the vertical line test.
- Determine whether a function is rational, algebraic, or trigonometric.
- Sketch the graphs of lines, parabolas, the basic trigonometric functions, and functions involving absolute values.
- Sketch the graphs of circles, ellipses, and hyperbolas.
- Solve simple equations (linear, quadratic, and trigonometric), and simple inequalities (including inequalities involving absolute values).
- Determine the equation of a line using the slope-intercept form and the point-slope form.
- Convert degrees into radians and vice versa.
- Recall the definition of all trigonometric ratios and use them to determine all trigonometric ratios from a given trigonometric ratio.
- Recall the trigonometric ratios of special angles on the unit circle.
- Determine the angle corresponding to a given trigonometric ratio.

1.1 Algebra

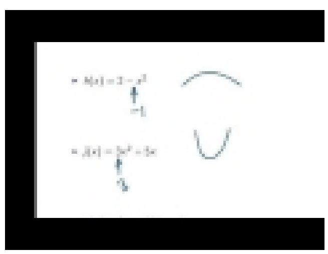
1.1.1 Things to know

- If $ax^2 + bx + c = 0$, the roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; the sum of the roots is $-b/a$; the product of the roots is c/a.
- $x^{-r} = \dfrac{1}{x^r}$.
- $x^r x^s = x^{r+s}$.
- $x^{1/n} = \sqrt[n]{x}$, for n integer. If n is even, then x must be non-negative and $\sqrt[n]{x}$ denotes the non-negative root of x.
- $(x^r)^s = x^{r \cdot s}$, assuming that $x \geq 0$. Note that care must be taken if x is negative. For instance, $(x^2)^{1/2} \neq x$ if $x < 0$; rather, $(x^2)^{1/2} = |x|$.
- Finally, avoid common mistakes: remember that, in general,

$$\sqrt{x+y} \neq \sqrt{x} + \sqrt{y},$$
$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y},$$
$$(x+y)^2 \neq x^2 + y^2.$$

1.1.2 Review videos

The following review videos may be useful:

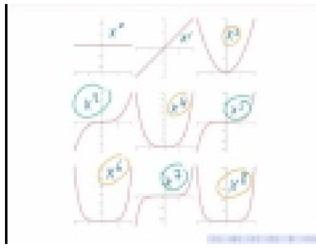


YouTube: `https://www.youtube.com/watch?v=huW3DPsY9VY`

Figure 1.1.1 A review video on quadratic functions

YouTube: `https://www.youtube.com/watch?v=VT8E3HTWJpM`

Figure 1.1.2 A review video on completing squares and zeros of quadratic functions



YouTube: `https://www.youtube.com/watch?v=ryzROWyVgOM`

Figure 1.1.3 A review video on power functions

1.2 Functions

1.2.1 Things to know

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

- **Domain and range:** The set D is called the **domain** of f, while the **range** of f is the set of all possible values of $f(x)$ as x varies through the domain.
- **Graph of a function:** all points in the xy-plane such that $y = f(x)$ with x in the domain of f.
- **Vertical line test:** A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.
- **Piecewise defined functions:** Functions that are defined using different formulae for different parts of their domains.
- **Absolute value function:** $f(x) = |x|$ is defined by

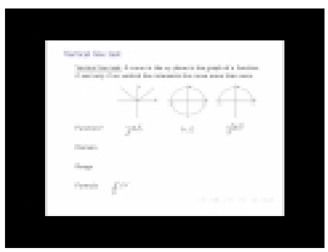
$$|x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}.$$

- **Increasing and decreasing functions:** A function is **increasing** (resp. **decreasing**) on some interval I if $f(x_1) < f(x_2)$ (resp. $f(x_1) > f(x_2)$) for all $x_1, x_2 \in I$ such that $x_1 < x_2$ (resp. $x_1 > x_2$).
- **Odd and even functions:** A function f such that $f(x) = f(-x)$ is **even**, while if $f(x) = -f(-x)$ it is **odd**.
- **Transformations of functions:**
  - Vertical shift: $y = f(x) + k$,
  - Horizontal shift: $y = f(x - k)$,
  - Vertical stretch: $y = cf(x)$,
  - Horizontal stretch: $y = f(x/c)$,
  - Reflection about the x-axis: $y = -f(x)$,
  - Reflection about the y-axis: $y = f(-x)$,
  - Composition of functions: $(f \circ g)(x) = f(g(x))$. Note that the domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

- **Types of functions:**
    - **Linear function:** $f(x) = mx + b$. Its graph is a line with slope m and y-intercept b.
    - **Polynomial function:** $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. The **degree** of $f(x)$ is n. When $n = 2$, $f(x)$ is called **quadratic**, while it is called **cubic** if $n = 3$.
    - **Power function:** $f(x) = x^a$. If a is a positive integer, then $f(x)$ is a particular example of a polynomial function. If $a = 1/n$ with n a positive integer, then $f(x)$ is a **root function** (for example, for $a = 1/2$ it is the familiar square root function $f(x) = x^{1/2} = \sqrt{x}$). For $a = -1$, it is the **reciprocal function** $f(x) = 1/x$.
    - **Rational function:** $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions.
    - **Algebraic function:** A function that can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with a polynomial function. All rational functions are clearly algebraic.

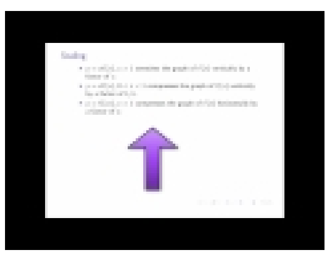
1.2.2 Review videos

The following review videos may be useful:



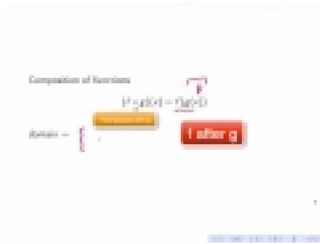
YouTube: `https://www.youtube.com/watch?v=onmTUoOu8sE`

Figure 1.2.1 A review video on functions



YouTube: `https://www.youtube.com/watch?v=Y0cqY8Zf6AQ`

Figure 1.2.2 A review video on transformations of functions



YouTube: `https://www.youtube.com/watch?v=90I1XKJV8tI`

Figure 1.2.3 A review video on composition of functions

1.3 Analytic geometry

1.3.1 Things to know

- **Lines:**
  - Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the xy-plane,

    $$\Delta x = x_2 - x_1 \qquad \text{is called the change in x, or the "run"},$$
    $$\Delta y = y_2 - y_1 \qquad \text{is called the change in y, or the "rise".}$$

    If $x_1 \neq x_2$, the line passing through the points P_1 and P_2 is non-vertical, and its slope is

    $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

  - If $P_1 \neq P_2$ but $\Delta y = 0$ then $m = 0$ and the line is **horizontal**. If $P_1 \neq P_2$ but $\Delta x = 0$ then the line is vertical, and its slope is undefined. A vertical line is not the graph of a function, since it does not pass the vertical line test.
  - The equation of a line has the form $y = mx + b$, where m is the **slope** and b is the y**-intercept**.
  - Two lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are **parallel** if $m_1 = m_2$. They are **perpendicular** if $m_1 = -1/m_2$.
  - To find the equation of the line with a given slope $m = a$ passing through a point $P(x_1, y_1)$, we can use the **point-slope formula**:

    $$y - y_1 = m(x - x_1).$$

    We expand and gather terms to get an equation of the form $y = mx + b$.
  - To find the equation of the line passing through two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we first determine that the slope of the line is

    $$m = \frac{y_2 - y_1}{x_2 - x_1},$$

    and then substitute into the point-slope formula to get

    $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

    We expand and gather terms to get an equation of the form $y = mx + b$.

  - The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- **Triangles:** The area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h the height.
- **Circles:**
  - The equation of a circle with centre (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

  - The area of a circle is $A = \pi r^2$.
  - The circumference of a circle is $C = 2\pi r$.
  - The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$ where θ is the angle in radians.
  - The length of an arc is $L = \theta r$.
- **Parabolas:** A parabola is the graph of a function of the form $y = ax^2 + bx + c$.
- **Ellipses:** The equation of an ellipse with centre (h, k) is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

- **Hyperbolas:** The equation of a hyperbola with "centre" (h, k) is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1.$$

- **Three-dimensional objects:**
  - The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.
  - The surface area of a sphere is $A = 4\pi r^2$.
  - The volume of a cylinder is $V = \pi r^2 h$ where r is the radius and h the height.
  - The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$.

1.4 Trigonometry

1.4.1 Things to know

- To convert angles between degrees and radians: $180^o = \pi$ radians.
- A point (x, y) on the unit circle at angle θ with respect to the positive side of the x-axis (positive angle meaning counterclockwise rotation) has coordinates $(x, y) = (\cos\theta, \sin\theta)$.
- Given a right triangle,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

- The four other trig functions can be obtained from $\sin\theta$ and $\cos\theta$ as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}.$$

- Some useful trigonometric identities:

  ◦
  $$\sin^2 A + \cos^2 A = 1, \qquad \tan^2 A + 1 = \sec^2 A,$$
  $$1 + \cot^2 A = \csc^2 A.$$

  Note that you only need to remember the first one, you can derive the other two from it.

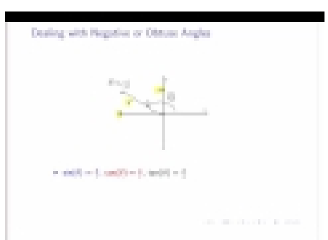
  ◦
  $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$
  $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

  ◦
  $$\sin 2A = 2 \sin A \cos A,$$
  $$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

  ◦
  $$\sin^2 A = \frac{1}{2}(1 - \cos 2A), \qquad \cos^2 A = \frac{1}{2}(1 + \cos 2A).$$

1.4.2 Review videos

The following review video may be useful:



YouTube: `https://www.youtube.com/watch?v=weUyNxzA5LM`

Figure 1.4.1 A review video on trigonometry

Chapter 2

A preview of calculus

2.1 A preview of differential and integral calculus from kinematics

We start with a preview of what differential and integral calculus is all about. We start from the point of view of kinematics. We study how differential calculus, and the notion of derivative, is intimately connected to the ideas of rate of change and velocity. We then look at integral calculus, and the notion of integral, and see how it relates to the ideas of area and distance. In both cases, we realize that the notion of "limit" is key. Those are the foundations of calculus, which we will study throughout this course.

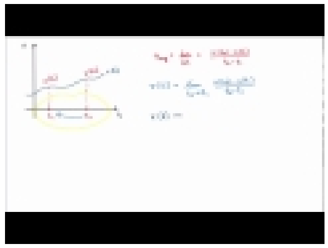
We are deliberately imprecise in this section: the aim is simply to give an overview of the foundational concepts of calculus, which we will develop more rigorously during the semester.

Objectives

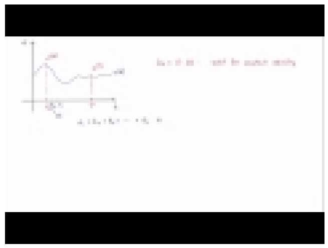
You should be able to:

- Explain the difference between average and instantaneous velocity.
- Describe the limit process that arises in the computation of an instantaneous velocity.
- Describe and illustrate how to approximate the area under a curve using approximating rectangles.
- Describe the limit process that arises in the computation of the area under a curve.
- Relate the calculation of the area under a curve to the calculation of the distance covered in kinematics.

2.1.1 Instructional videos



YouTube: `https://www.youtube.com/watch?v=LzwagSBAUYY`



YouTube: `https://www.youtube.com/watch?v=h3qphnxINIE`

2.1.2 Key concepts

Concept 2.1.1 Derivative, rate of change and velocity. Given a function f:

- The **average rate of change** of f over the interval $[x_1, x_2]$ is given by

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

- The **instantaneous rate of change** of f at $x = c$ is called the **derivative** of f at $x = c$ and is denoted by $f'(c)$ or $\left.\frac{df}{dx}\right|_{x=c}$. It is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

  The limit here means that we take the average rate of change over an interval $[x, c]$ with c as close as possible to x.

- If the function f is the position function of an object, then its average rate of change is the **average velocity** of the object over a time interval, while its instantaneous rate of change is its **instantaneous velocity**.

Concept 2.1.2 Integral, distance and area. Let $v(t)$ be the velocity function of an object, and assume that it is always positive (for simplicity):

- The distance covered by the object between t_1 and t_2 can be obtained by calculating the area under the graph of $v(t)$ between $t = t_1$ and $t = t_2$.

- An approximation can be obtained by slicing the area into rectangles of width Δt and summing over the areas of the rectangles.

- By taking the limit $\Delta t \to 0$ (equivalently, by sending the number of rectangles to infinity), we obtain an exact expression for the area under the graph and the distance covered. The resulting expression is called the **integral of the function $v(t)$ between $t = t_1$ and $t = t_2$**, and is written as

$$\int_{t_1}^{t_2} v(t)\ dt.$$

It is the "inverse operation of differentiation", as we will see later on when we study the Fundamental Theorem of Calculus.

2.1.3 Further readings

Section 1.2 in CLP1

2.2 Tangent lines and derivatives

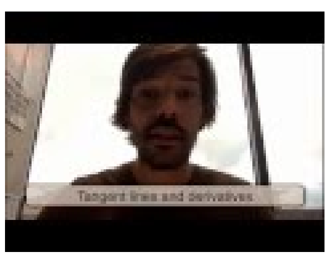
In the previous section we introduced the concept of derivative through kinematics. Here we see that the derivative also has a deep interpretation in geometry, as calculating the slope of the tangent line to the graph of a function. We study our first "tangent line problem", which leads us to the question of how to evaluate (and define) limits.

Objectives

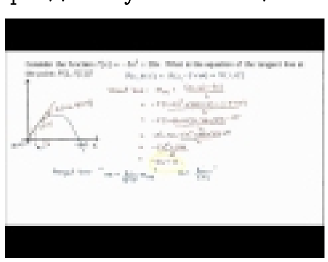
You should be able to:

- Describe and illustrate the connection between the velocity and tangent problems.
- Explain the difference between the slope of a secant line connecting two points on a curve and the slope of the tangent line to a curve at a point.
- Describe the limit process that arises in the calculation of the slope of a tangent line.
- Use correct notation for the limit process, to represent the slope of a tangent line as an appropriate limit of the slope of a secant line.
- Calculate the equation of the tangent line for simple functions.

2.2.1 Instructional videos



YouTube: `https://www.youtube.com/watch?v=8DNxR8jW2SU`



YouTube: `https://www.youtube.com/watch?v=t6oAlqvUHjw`

2.2.2 Key concepts

Concept 2.2.1 Secant line. Given two points $P_1(c, f(c))$ and $P_2(x_2, f(x_2))$ on the graph of a function f, the **slope of the secant line through P_1 and P_2** is given by

$$m_{P_1 P_2} = \frac{f(x_2) - f(c)}{x_2 - c}.$$

Equivalently, defining $x_2 = c + h$, it can be written as

$$m_{P_1 P_2} = \frac{f(c+h) - f(c)}{h}.$$

This is known as the **difference quotient** of the function f at $x = c$.

Concept 2.2.2 Tangent line. The **slope of the tangent line to the curve $y = f(x)$ at $P_1(c, f(c))$** is obtained by taking the limit $x_2 \to c$ or, equivalently, $h \to 0$:

$$\begin{aligned} m &= \lim_{x_2 \to c} \frac{f(x_2) - f(c)}{x_2 - c} \\ &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \end{aligned}$$

Thus, m is precisely equal to the derivative of f at $x = c$; that is, $m = f'(c)$.

2.2.3 Further readings

Section 1.1 in CLP1

Chapter 3

Limits

3.1 An informal definition of limits

Our brief overview of calculus convinced us that the study of limits is foundational in calculus. In this section we give an informal definition of limits, and show how the value of a limit can (sometimes, but not always) be estimated using tables of values. We also define the notion of one-sided limits.

Objectives

You should be able to:

- Explain and illustrate the concept of the limit of a function $f(x)$ as x approaches a.
- Construct examples of limits that exist and examples of limits that do not exist.
- Identify a given limit (or conclude that it does not exist) from the graph of a function.
- Explain the difference between left- and right-sided limits.
- Relate limits to one-sided limits.
- Estimate the value of a limit using a table of values.

3.1.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=07s1XkBToeU`

3.1.2 Key concepts

Concept 3.1.1 The informal definition of limits. We write

$$\lim_{x\to a} f(x) = L,$$

and say that ***the limit of*** $f(x)$***, as*** x ***approaches*** a***, is equal to*** L, if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a (on either side of a) but not equal to a.

Note that saying that $\lim\limits_{x\to a} f(x) = L$ is ***not the same*** as saying that $f(a) = L$. The former is about the behaviour of $f(x)$ for x near a, while the latter is the value of the function $f(x)$ at the point $x = a$.

Concept 3.1.2 Estimating the value of a limit using a table of values. We can often estimate $\lim\limits_{x\to a} f(x)$ by writing down a table of values for the function $f(x)$ with x closer and closer to a (from both sides).

Concept 3.1.3 One-sided limits. We write

$$\lim_{x\to a^-} f(x) = L,$$

if $f(x)$ gets close to L when x approaches a ***from the left*** (from ***below***), and

$$\lim_{x\to a^+} f(x) = L,$$

if $f(x)$ gets close to L when x approaches a ***from the right*** (from ***above***).

Concept 3.1.4 Limits vs one-sided limits. Note that

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = L.$$

3.1.3 Further readings

Section 1.3 in CLP1

3.2 The formal definition of limits

The informal definition of limits given in the previous subsection is imprecise, since we have not defined what it means to say “arbitrarily close” and “sufficiently close”. The formal definition of limits makes such statements precise.

Objectives

You should be able to:

- State the formal definition of limits.

3.2.1 Instructional video

There is no instructional video on this particular topic. It will be covered during the live lecture.

3.2.2 Key concepts

Concept 3.2.1 The formal definition of limits. Let $a \in \mathbb{R}$, and let f be a function defined on some open interval that contains $x = a$, except possibly at $x = a$ itself. Then we write

$$\lim_{x \to a} f(x) = L,$$

and say that the ***limit of*** $f(x)$***, as*** x ***approaches*** a***, is*** L, if and only if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that:

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Since ϵ is an arbitrary positive number, what this says is that we can make the distance between $f(x)$ and L arbitrarily small, by taking the distance between x and a sufficiently small.

3.2.3 Further readings

Section 1.7 in CLP1

3.3 Infinite limits and vertical asymptotes

So far we have studied limits that "exist", in the sense that as x approaches a, the value of the function $f(x)$ gets arbitrarily close to a finite value L. But there is another interesting case: sometimes the value of $f(x)$ becomes arbitrarily large (either positive or negative) as $x \to a$. This is the concept of "infinite limits", which is what we study in this section. We also study the related concept of vertical asymptotes.

Objectives

You should be able to:

- Evaluate infinite limits.
- Explain and illustrate the connection between infinite limits and vertical asymptotes.
- Determine whether a limit exists or not.
- Determine the vertical asymptotes of the graph of a function.

3.3.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=t58_ahWlIB8`

3.3.2 Key concepts

Concept 3.3.1 Infinite limits. Let $f(x)$ be a function that is defined near $x = a$ (but not necessarily at $x = a$).

We write

$$\lim_{x \to a} f(x) = \infty,$$

and say that ***the limit of*** $f(x)$***, as*** x ***approaches*** a***, is infinity*** if the values of $f(x)$ can be made arbitrarily large and positive by taking x sufficiently close to a (on either side of a) but not equal to a.

Similarly, we write

$$\lim_{x \to a} f(x) = -\infty,$$

and say that ***the limit of*** $f(x)$***, as*** x ***approaches*** a***, is negative infinity*** if the values of $f(x)$ can be made arbitrarily large and negative by taking x sufficiently close to a (on either side of a) but not equal to a.

Concept 3.3.2 Vertical asymptotes. The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or both.}$$

Note that the function $f(x)$ does not have to blow up on both sides of $x = a$ for it to be a vertical asymptote; as long as the limit is infinite on one side of $x = a$ it is a vertical asymptote.

Concept 3.3.3 Existence of limits. We say that $\lim_{x \to a} f(x)$ **exists** if $\lim_{x \to a} f(x) = L$, with L a ***finite number*** ($L = 0$ is perfectly fine).

So there are two reasons why a limit may not exist:

1. The left-sided and right-sided limits may be different:

   $$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x),$$

   in which case we write that

   $$\lim_{x \to a} f(x) \text{ DNE.}$$

2. The limit may be infinite. In this case we write

   $$\lim_{x \to a} f(x) = \pm\infty,$$

   keeping in mind that the limit does not exist since $\pm\infty$ is not a finite number.

3.3.3 Further readings

Section 1.3 in CLP1

3.4 How to evaluate limits

Well, now that we know everything about limits, why not evaluate them? Let us now study various techniques to evaluate limits!

Objectives

You should be able to:

- Evaluate a given limit (or conclude that it does not exist) using the limit laws and algebraic manipulations such as factoring, making a common denominator and/or rationalizing the numerator or denominator.
- For the limit of a quotient function $f(x)/g(x)$, distinguish between three possible outcomes (L, "$A/0$" where A is not equal to 0, and "0/0").
- Explain, illustrate and apply the Squeeze Theorem.

3.4.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=xdq7x-HjceA`

3.4.2 Key concepts

Concept 3.4.1 Limit laws. Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

- **Limit of a constant:** $\lim_{x\to a} c = c$ for any $c \in \mathbb{R}$.
- **Limit of x:** $\lim_{x\to a} x = a$.
- **Sum rule:** $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.
- **Product rule:** $\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$.
- **Quotient rule:** $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.
- **Root rule:** $\lim_{x\to a} (f(x))^{1/n} = \left(\lim_{x\to a} f(x)\right)^{1/n}$, where n is a positive integer. For n even we require that $\lim_{x\to a} f(x) > 0$ so that the root is real and the limit is well defined.

Concept 3.4.2 Simplifying a function. If $f(x) = g(x)$ for $x \neq a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x),$$

provided the limits exist. In practice, what this means is that when you are evaluating $\lim_{x\to a} f(x)$, then you can manipulate $f(x)$ by simplifying it using things like factorization, rationalization, common denominator, etc., assuming that $x \neq a$, and it will not change the limit.

Concept 3.4.3 Flowchart. This flowchart is for calculating the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

for simple enough functions $f(x)$ and $g(x)$ (not piecewise-defined, no absolute values, etc.) such that both $f(a)$ and $g(a)$ are finite. This is the form of many functions that we will encounter for the time being. If your function is not of this form, you may first need to use algebraic techniques, such as common denominator, to transform it into this form.

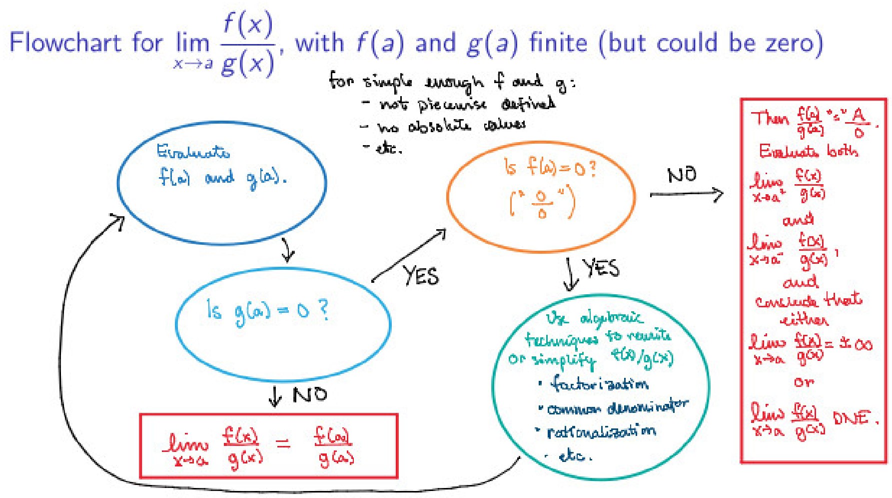


Figure 3.4.4 Flowchart for evaluating limits

Concept 3.4.5 Squeeze theorem. First, if $f(x) \leq g(x)$ for all x near a (except possibly at $x = a$), and if the limits of both functions exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

The **Squeeze theorem** states that, if $g(x) \leq f(x) \leq h(x)$ for all x near a (except possibly at $x = a$), and if

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L, \qquad \text{then} \qquad \lim_{x \to a} f(x) = L.$$

3.4.3 Further readings

Section 1.4 of CPL1

3.5 Continuity

We encountered some functions whose values become arbitrarily large as $x \to a$. Those are examples of functions that are not "continuous". Roughly speaking, a function is continuous if you can draw the graph of the function without lifting your pen. In this section we study the notion of continuity in more detail.

Objectives

You should be able to:

- Explain and illustrate the definition of continuity.
- Explain and illustrate infinite, jump, and removable discontinuities.
- Apply the definition of continuity to determine whether or not a function is continuous at a point.

3.5.1 Instructional video

There is no instructional video on this particular topic. It will be covered during the live lecture.

3.5.2 Key concepts

Concept 3.5.1 Continuity. A function $f(x)$ is **continuous at** $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

Thus, for a function to be continuous, it must satisfy the three conditions:

1. $f(a)$ exists;
2. $\lim_{x\to a} f(x)$ exists;
3. $\lim_{x\to a} f(x) = f(a)$.

A function is **continuous on an interval** if it is continuous at every point in the interval. Practically, it is continuous over an interval if you can draw the graph of the function over this interval without lifting your pen.

A function is **continuous from the right at** $x = a$ if

$$\lim_{x\to a^+} f(x) = f(a),$$

while it is **continuous from the left at** $x = a$ if

$$\lim_{x\to a^-} f(x) = f(a),$$

Concept 3.5.2 Combining continuous functions. Let $a, c \in \mathbb{R}$, and $f(x), g(x)$ be functions that are continuous at $x = a$. Then the following functions are also continuous at $x = a$:

$$\begin{aligned}
&f(x) + g(x),\\
&f(x) - g(x),\\
&cf(x),\\
&f(x)g(x),\\
&\frac{f(x)}{g(x)} \qquad \text{provided } g(a) \neq 0.
\end{aligned}$$

Concept 3.5.3 Some continuous functions. The following functions are continuous at every point in their domain: polynomials, rational functions, root functions, trigonometric functions, exponential functions, and logarithmic functions. Most functions that appear in physics are continuous.

Concept 3.5.4 Composition of continuity. Let $f(x)$ be a function that is continuous at $x = b$, and $\lim_{x\to a} g(x) = b$. Then

$$\lim_{x\to a} f(g(x)) = f(b) = f\left(\lim_{x\to a} g(x)\right).$$

Thus, if $g(x)$ is continous at $x = a$, we have that $\lim_{x\to a} g(x) = g(a)$. Therefore

$$\lim_{x\to a} f(g(x)) = f(g(a)),$$

that is, the composition

$$(f \circ g)(x) = f(g(x))$$

is continuous at $x = a$.

3.5.3 Further readings

Section 1.6 in CLP1

3.6 Limits at infinity and horizontal asymptotes

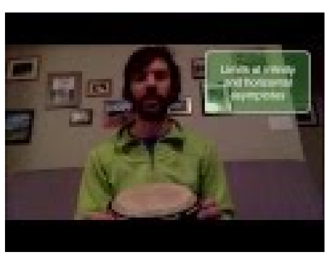
We finally introduce one last type of limits, namely limits at infinity, and the related concept of horizontal asymptotes. These concern the behaviour of functions when you take the argument to be arbitrarily large (either positive or negative).

Objectives

You should be able to:

- Evaluate limits at infinity.
- Explain and illustrate the connection between limits at infinity and horizontal asymptotes.
- Find the horizontal asymptotes, if any, of a function.

3.6.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=nlfLXbGAeEU`

3.6.2 Key concepts

Concept 3.6.1 Limits at infinity. Let $f(x)$ be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

if the values of $f(x)$ can be made ***arbitrarily close to*** L ***by requiring*** x ***to be sufficiently large***.

Similarly, for a function $f(x)$ defined on some interval $(-\infty, a)$,

$$\lim_{x\to-\infty} f(x) = L$$

if the values of $f(x)$ can be made ***arbitrarily close to*** L ***by requiring*** x ***to be sufficiently large but negative***.

In other words, we are looking at whether the function converges to a finite number L as x becomes very large either on the positive or negative side.

Note that this definition can be generalized to include limits at infinity that are infinite:

$$\lim_{x\to\infty} f(x) = \infty$$

means that the values of $f(x)$ can be made ***arbitrarily large for*** x ***sufficiently large***. A similar definition holds for $-\infty$.

It can also happen that limits at infinity DNE, such as $\lim_{x\to\infty} \sin(x)$.

Concept 3.6.2 Some useful limits. For all rational numbers $r > 0$,

$$\lim_{x\to\infty} \frac{1}{x^r} = 0,$$

and for all rational numbers $r > 0$ such that x^r is well defined for all x,

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Concept 3.6.3 Tip to evaluate limits at infinity for rational functions. To evaluate limits at infinity for rational functions (it also works for ratios of functions involving roots), divide both numerator and denominator by the largest power of x that occurs in the denominator.

Concept 3.6.4 Horizontal asymptotes. The horizontal line $y = L$ is a ***horizontal asymptote*** of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L, \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

3.6.3 Further readings

Section 1.5 of CPL1

Chapter 4

Differentiation

4.1 The derivative of a function

In our overview of calculus, we realized that the derivative of a function was a fundamental concept: given the position function of an object, it ouputs its velocity function. Generally, it gives the instantaneous rate of change of a function. Geometrically, the derivative of a function at a point gives the slope of the tangent line to the graph of the function at that point.

We have also seen that the derivative of a function is defined as a limit. Now that we know how to evaluate limits, we can go back to the concept of derivatives, and study it in more depth!

Objectives

You should be able to:

- Explain why the derivative of a function is itself a function.
- Calculate the derivative of simple functions (such as linear and quadratic polynomials, square root function, and simple rational functions) from the definition of the derivative of a function.
- Sketch the graph of the derivative of a function from the graph of the function itself.

4.1.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=qdiI1fzv2r8`

4.1.2 Key concepts

Concept 4.1.1 The derivative of a function. The **derivative of a function** $f(x)$ is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

We can calculate the derivative of any function directly from the definition, by evaluating the limit above.

Concept 4.1.2 Notation. The following are equivalent notations for the derivative of a function $y = f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x).$$

Concept 4.1.3 Interpretation of the derivative. The value of $f'(x)$ of $x = a$ represents:

- The instantaneous rate of change of $f(x)$ at $x = a$ (instantaneous velocity if f is a position function);
- The slope of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$.

Concept 4.1.4 Higher derivatives.

- The **second derivative** of f is denoted by $f''(x)$ or

  $$\frac{d^2 f}{dx^2}.$$

  It is the derivative of f', that is, the derivative of the derivative of f. For instance, if f is a position function, f' is the velocity function and f'' is the acceleration function.

- Similarly, the **third derivative** of f is denoted by $f'''(x)$ or

  $$\frac{d^3 f}{dx^3}.$$

  Is it the derivative of f''.

- In general, the n**'th derivative** of f is denoted by $f^{(n)}(x)$ or

  $$\frac{d^n f}{dx^n}.$$

  It is the derivative of $f^{(n-1)}$, that is, it is obtained from f by differentiating n times.

4.1.3 Further readings

Section 2.2 in CLP1Section 2.3 in CLP1Section 2.14 in CLP1

4.2 Differentiability

In the previous section we reviewed that the derivative of a function is defined as a limit. But when we studied limits, we realized that sometimes limits do not exist. What happens if the limit in the definition of the derivative of a given function does not exist at a point? This is what we study in this section. Basically, a function $f(x)$ is "differentiable" at a point $x = a$ if the limit in the definition of its derivative $f'(a)$ exists. Let us now study this in more detail!

Objectives

You should be able to:

- Explain and illustrate the definition of differentiability.
- Apply the definition of differentiability to determine whether or not a function is differentiable at a point a.
- Visually identify where functions are not differentiable (corners, discontinuities, vertical tangents), and explain why.
- Relate the notions of differentiability and continuity.

4.2.1 Instructional video

There is no instructional video on this particular topic. It will be covered during the live lecture.

4.2.2 Key concepts

Concept 4.2.1 Differentiability. A function f is **differentiable** at $x = a$ if $f'(a)$ exists (as a limit). It is differentiable on an open interval (a, b) if it is differentiable at every point in the interval.

Concept 4.2.2 Relation between continuity and differentiability. A function f that is not continuous at $x = a$ is also not differentiable at $x = a$. Equivalently, a function f that is differentiable at $x = a$ must be continuous at $x = a$.

The converse is ***not true***: a function f that is continuous at $x = a$ may not be differentiable at $x = a$ (think of $f(x) = |x|$ at $x = 0$).

Concept 4.2.3 Where functions fail to be differentiable. A function can fail to be differentiable at $x = a$ in three different ways:

- It is ***not continuous*** at $x = a$;
- The left-sided limits and right-sided limits of the difference quotient are not the same, in which case f has a ***corner*** or a ***kink*** at $x = a$;
- The limit of the difference quotient is infinite, in which case f has a ***vertical tangent*** at $x = a$.

4.2.3 Further readings

Section 2.2.3 in CLP1

4.3 Differentiation rules

Even though in principle we can calculate the derivative of any differentiable function from its definition as the limit of the difference quotient, in practice such limit calculations quickly become rather painful. We need to find a better way of calculating derivatives. This is what we study in this section and in the next few sections. We prove a number of rules, collectively called "differentiation rules", which allow us to drastically simplify the calculation of derivatives of complicated functions. We prove these rules using the definition of the derivative as a limit: but once the rules are proven, we can use them directly to calculate derivatives of complicated functions.

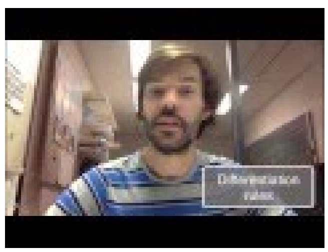
We start in this section by studying the following foundational differentiation rules: the power rule, the constant multiple rule, the sum and difference rules, the product rule, and the quotient rule.

Objectives

You should be able to:

- Derive foundational rules of differentiation (such as power rule, constant multiple rule, sum rule, product rule, quotient rule) from the definition of the derivative of a function.
- Recall foundational rules of differentiation (such as power rule, constant multiple rule, sum rule, product rule, quotient rule), and use them to calculate derivatives.

4.3.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=jH3Jqz5-j4U`

4.3.2 Key concepts

Concept 4.3.1 Power rule. For any real number $a \in \mathbb{R}$,

$$\frac{d}{dx}\left(x^a\right) = ax^{a-1}.$$

Note that it follows from the power rule that the derivative of any constant c is always zero.

Concept 4.3.2 Constant multiple rule. For any constant c and differentiable function f,

$$\frac{d}{dx}\left(cf(x)\right) = cf'(x).$$

Concept 4.3.3 Sum and difference rules. For any two differentiable functions f and g,

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x).$$

Concept 4.3.4 Product rule. For any two differentiable functions f and g,

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

Note that this is ***not*** equal to the product of the derivatives $f'(x)g'(x)$.

Concept 4.3.5 Quotient rule. For any two differentiable functions f and g,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}.$$

Note that this is ***not*** equal to the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$

An easy way to remember the quotient rule is to sing the song:

low d-hi minus hi d-low, draw the line and square below!

Finally, note that the quotient rule is actually a consequence of the product rule and the chain rule, as we will see shortly.

4.3.3 Further readings

Section 2.4 in CLP1Section 2.5 in CLP1Section 2.6 in CLP1

4.4 Derivatives of trigonometric functions

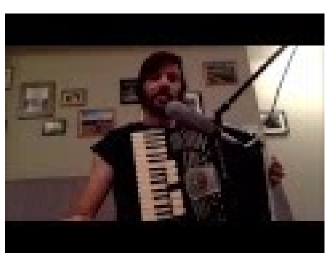
In the previous section we proved a few foundational differentiation rules from the definition of the derivative. In this section we use the definition of the derivative and differentiation rules to calculate the derivatives of trigonometric functions. We encounter along the way a few interesting and non-trivial limits involving trigonometric functions.

Objectives

You should be able to:

- Determine the derivatives of the primary trigonometric functions, $\sin x$ and $\cos x$, from the definition of derivatives, and recall the result.
- Calculate the derivatives of other trigonometric functions using the derivatives of the primary trigonometric functions, the quotient rule and trigonometric identities.
- Evaluate certain limits involving trigonometric functions, such as $\lim_{x\to 0}\frac{\sin x}{x}$ and $\lim_{x\to 0}\frac{\cos x - 1}{x}$.

4.4.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=DxG7qouILuw`

4.4.2 Key concepts

Concept 4.4.1 Useful trigonometric limits.

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

The first one can be proved using the Squeeze Theorem; the second one then follows using trigonometric identities and limit laws.

Concept 4.4.2 Derivatives of trigonometric functions.

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$
$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x,$$
$$\frac{d}{dx}\sec x = \sec x\tan x, \qquad \frac{d}{dx}\csc x = -\csc x\cot x.$$

The first two can be proved from the definition of the derivative; the other ones then follow using the quotient rule.

4.4.3 Further readings

Section 2.8 in CLP1

4.5 Chain rule

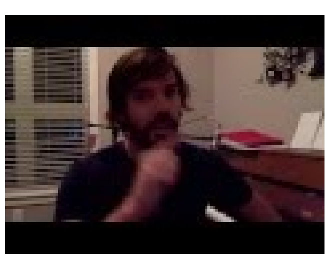
We now study another differentiation rule, known as the "chain rule". It is useful to calculate the derivative of a "composite function", namely a "function of a function". For instance, if f and g are differentiable functions, then the chain rule can be used to calculate the derivative of the composite function $F(x) = f(g(x))$.

Objectives

You should be able to:

- Recall the statement of the chain rule and identify when it is required to calculate the derivative of function.
- Apply the chain rule to calculate derivatives of functions.
- Translate between different notations for the chain rule.

4.5.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=sbew3V0nxpI`

4.5.2 Key concepts

Concept 4.5.1 The chain rule. Let f and g be two functions such that g is differentiable at x and f is differentiable at $g(x)$. Then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x, and its derivative is given by

$$F'(x) = f'(g(x)) \cdot g'(x).$$

This is known as the **chain rule**.

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then the chain rule can be written as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

A tip: When using the chain rule, you work from the outside to the inside. You first differentiate the outer function f (evaluated at the inner function $g(x)$) and then multiply by the derivative of the inner function.

Concept 4.5.2 Do we need the quotient rule? We derived previously the quotient rule. But do we really need it? Instead of using the quotient rule, you can always use the product rule and the chain rule instead. Indeed,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)(g(x))^{-1}\right),$$

and then you can calculate the right-hand-side by using the product rule and the chain rule.

4.5.3 Further readings

Section 2.9 in CLP1

4.6 Implict differentiation

So far we have mostly dealt by functions that are given "explicitly", as $y = f(x)$. But sometimes functions are defined "implicitly": this happens when a function $y = f(x)$ is defined implicitly by a relation between y and x (which you may or may not be able to solve explicitly for y). We study such implicit functions in this section, and show how we can calculate their derivatives. The resulting process is known as "implicit differentiation".

Objectives

You should be able to:

- Explain and illustrate the concept of implicit functions.
- Use the method of implicit differentiation to calculate the derivative of implicit functions.

4.6.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=fL7_6DrZcc0`

4.6.2 Key concepts

Concept 4.6.1 Implicit functions. Given a relation

$$H(x, y) = 0,$$

we say that f is a function defined implicitly by this relation if

$$H(x, f(x)) = 0$$

for all x in the domain of f.

A relation $H(x, y) = 0$ defines a curve in the xy-plane, which implicitly defines y as one or several functions of x.

Concept 4.6.2 Implicit differentiation. Implicit differentiation is the process of calculating the derivative of a function defined implicitly by a relation $H(x, y) = 0$.

To calculate the derivative y' of an implicit function:

1. Treat the variable y in the relation as an unknown but differentiable function of x (like $y = g(x)$), and differentiate both sides of the relation with respect to x, using the chain rule.

2. Collect the terms involving y' on one side of the equation and solve for y'.

Note that this will generally give y' as a function of x and y, where y is understood as the function implicitly defined by the original relation.

4.6.3 Further readings

Section 2.11 in CLP1

4.7 Inverse functions

In this section we review the concept of inverse functions, and use implicit differentiation to calculate the derivative of an inverse function. This will helpful to study exponentials, logarithms, and inverse trigonometric functions in the next few sections.

Objectives

You should be able to:

- Determine whether a given function has an inverse.

- Relate the graph of a function and of its inverse.
- Relate the domain and the range of a function and of its inverse.
- Calculate explicitly the inverse of a function when possible.
- Calculate the derivative of an inverse function using implicit differentiation.

4.7.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=I1Qjo_WMWqk`

4.7.2 Key concepts

Concept 4.7.1 One-to-one functions. A function is **one-to-one** (or **injective**) if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2.$$

Given the graph of a function, it is easy to see whether it is one-to-one: a function is one-to-one if no horizontal line intersects its graph more than once. This is sometimes known as the **horizontal line test**.

Concept 4.7.2 Inverse functions. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$y = f(x) \qquad \text{if and only if} \qquad f^{-1}(y) = x.$$

The domain of f is the range of f^{-1}, while the range of f is the domain of f^{-1}.

Inverse functions satisfy:

$$\begin{aligned} f^{-1}(f(x)) &= x \qquad \text{for } x \text{ in the domain of } f, \\ f(f^{-1}(x)) &= x \qquad \text{for } x \text{ in the domain of } f^{-1}. \end{aligned}$$

The graph of $y = f^{-1}(x)$ can be obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

Note that $f^{-1}(x)$ is not the same as the reciprocal $[f(x)]^{-1} = \dfrac{1}{f(x)}$.

Concept 4.7.3 How to calculate f^{-1} from f.

1. Calculate the domain and the range of f (those will become the range and the domain of f^{-1}).
2. Check that f is a one-to-one function, or restrict its domain so that it is one-to-one on this restricted domain.

3. Write $y = f(x)$. Solve this equation for x in terms of y (if possible). This gives you the inverse function $x = f^{-1}(y)$ as a function of y.

4. To express f^{-1} as a function of x, interchange x and y, so that you get $y = f^{-1}(x)$.

Concept 4.7.4 The derivative of an inverse function. To calculate the derivative of an inverse function $y = f^{-1}(x)$, we use implicit differentiation. We know that:

$$y = f^{-1}(x) \qquad \text{if and only if} \qquad x = f(y).$$

We differentiate both sides of the relation $x = f(y)$ with respect to x, treating y as an unknown but differentiable function of x. We get:

$$1 = f'(y) \cdot \frac{dy}{dx}.$$

We solve for y' to get

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

4.7.3 Further readings

Section 0.6 in CLP1

4.8 Exponentials and logarithms

We now introduce exponentials and logarithms. Using what we learned about inverse functions and implicit differentiation, we calculate their derivatives. We also study logarithmic differentiation, which is a fancy application of implicit differentiation.

Objectives

You should be able to:

- Sketch the graphs of exponential and logarithmic functions, and recall their main properties.
- Derive the derivative of exponential and logarithmic functions from the definition of derivatives, inverse functions and implicit differentiation.
- Differentiate functions involving exponential and logarithmic functions.
- Differentiate functions involving products, quotients or powers by using logarithms (logarithmic differentiation).
- Find the number e as a limit.

4.8.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=uE-xbwImv-o`

4.8.2 Key concepts

Concept 4.8.1 Exponential functions. **Exponential functions** are functions of the form $f(x) = a^x$ for some positive constant a. The domain of $f(x) = a^x$, for any a, is $\mathbb{R}$, while the range (for $a \neq 1$) is $(0, \infty)$ (as a^x is always positive for any real number x).

For a and b positive numbers and x and y real numbers, exponential functions satisfy:

- $a^{x+y} = a^x a^y$,
- $a^{x-y} = \frac{a^x}{a^y}$,
- $(a^x)^y = a^{xy}$,
- $(ab)^x = a^x b^x$.

Concept 4.8.2 The natural exponential function. The base $e \simeq 2.71828$ is such that the slope of the tangent line to $y = e^x$ at $(0, 1)$ is exactly one. The function $f(x) = e^x$ is so cool that is has its own name: it is called the **natural exponential function**.

Concept 4.8.3 Logarithmic funcctions. The **logarithmic function with base** a, denoted by $f(x) = \log_a(x)$, is the inverse function of the exponential function a^x. That is,

$$x = a^y \qquad \text{if and only if} \qquad \log_a(x) = y.$$

The domain of logarithmic functions (for $a \neq 1$) is $(0, \infty)$, while the range is $\mathbb{R}$ (as it is the inverse of the exponential function a^x, and so the domain and range are exchanged).

By definition of inverse functions, we have:

$$\begin{aligned} \log_a(a^x) =& x \qquad \text{for } x \in \mathbb{R}, \\ a^{\log_a(x)} =& x \qquad \text{for } x \in (0, \infty). \end{aligned}$$

For x and y positive numbers, and r a real number, logarithmic functions satisfy:

- $\log_a(xy) = \log_a(x) + \log_a(y)$,
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$,
- $\log_a(x^r) = r \log_a(x)$.

Concept 4.8.4 The natural logarithm. Just as the natural exponential function e^x is very cool, so is its inverse. The logarithm with base e is called the **natural logarithm** and is denoted by $\ln(x) := \log_e(x)$.

Concept 4.8.5 Change of base formula. For any positive number $a \neq 1$,

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Concept 4.8.6 Derivatives of exponential and logarithmic functions.

$$\begin{aligned}
\frac{d}{dx}(e^x) &= e^x, \\
\frac{d}{dx}(a^x) &= a^x \ln(a), \\
\frac{d}{dx}(\ln(x)) &= \frac{1}{x}, \\
\frac{d}{dx}(\log_a(x)) &= \frac{1}{x \ln(a)}.
\end{aligned}$$

Concept 4.8.7 Logarithmic differentiation. Suppose that you are given a function $y = f(x)$. The idea of logarithmic differentiation is to "take the logarithm and then differentiate". More precisely, we first take the absolute value (as the argument of a logarithm must always be positive), and then take the natural logarithm on both sides of the relation to get

$$\ln|y| = \ln|f(x)|.$$

Note that if the function $f(x)$ is always positive, you can drop the absolute value. We then use implicit differentiation, i.e. we differentiate both sides with respect to x, considering that y is an arbitrary function of x. We get

$$\frac{y'}{y} = \frac{d}{dx} \ln|f(x)|.$$

We then solve for y', and substitute back $y = f(x)$, to get

$$y' = f(x) \frac{d}{dx} \ln|f(x)|.$$

We can get y' by evaluating the remaining derivative on the right-hand-side, for a specific choice of $f(x)$.

This method is useful when it is easier to evaluate the derivative of $\ln|f(x)|$ than of $f(x)$: for instance, for functions $f(x)$ such that both the base and the exponent depend on x (for instance, consider the function $f(x) = x^x$ with $x > 0$, in which case $\ln f(x) = \ln(x^x) = x \ln x$), or for functions $f(x)$ that are complicated products or quotients of functions (in which case logarithmic differentiation is faster than product and quotient rules).

4.8.3 Further readings

Section 2.7 in CLP1Section 2.10 in CLP1

4.9 Inverse trigonometric functions

Having studied the inverse functions to exponential functions, we now introduce the inverse functions to trigonometric functions, known as "inverse trigonometric functions". Their definition requires restricting the domain of trigonometric functions, to make them one-to-one (so that their inverse functions can be defined unambiguously). We also study their derivatives, using implicit differentiation.

Objectives

You should be able to:

- Determine the domain, the range and the graph of inverse trigonometric functions.
- Evaluate the value of inverse trigonometric functions at certain points.
- Calculate the derivative of inverse trigonometric functions using implicit differentiation.
- Simplify expressions involving trigonometric and inverse trigonometric functions.

4.9.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=5EgndoksPcM`

4.9.2 Key concepts

Concept 4.9.1 Inverse trigonometric functions.

We remark here that there is no universal convention for the choice of principal domain for $\sec x$ and $\csc x$ to define their inverse functions. For instance, in Section 2.12 of CLP1, a different choice is made (see Definition 2.12.4). The principal domain of $\sec x$ is chosen to be $x \in [0, \pi/2) \cup (\pi/2, \pi]$, while the principal domain of $\csc x$ is chosen to be $x \in [-\pi/2, 0) \cup (0, \pi/2]$. The domains of both inverse functions $\sec^{-1} y$ and $\csc^{-1} y$ remain $y \in (-\infty, -1] \cup [1, \infty)$.

Those choices may look simpler than ours, but there is a price to pay. With this choice of principal domains, the derivatives of the inverse secant and cosecant functions look a bit more complicated. Using implicit differentiation, we would then obtain:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}},$$

and

$$\frac{d}{dx} \csc^{-1} x = -\frac{1}{|x|\sqrt{x^2 - 1}}.$$

(See Theorem 2.12.8 in CLP1.) Note the appearance of absolute values here, which are not present with our choice of conventions.

In the end, it does not matter which choice of convention is used to define the inverse secant and cosecant functions. But one must be consistent. In this course we will use the choice of conventions presented in the main text, which results in derivatives without absolute values.

- Inverse sin function:
$$y = \sin x \qquad \Leftrightarrow \qquad x = \sin^{-1} y \qquad \text{for } x \in [-\pi/2, \pi/2], \; y \in [-1, 1].$$

- Inverse cos function:
$$y = \cos x \qquad \Leftrightarrow \qquad x = \cos^{-1} y \qquad \text{for } x \in [0, \pi], \; y \in [-1, 1].$$

- Inverse tan function:
$$y = \tan x \qquad \Leftrightarrow \qquad x = \tan^{-1} y \qquad \text{for } x \in (-\pi/2, \pi/2), \; y \in \mathbb{R}.$$

- Inverse cotan function:
$$y = \cot x \qquad \Leftrightarrow \qquad x = \cot^{-1} y \qquad \text{for } x \in (0, \pi), \; y \in \mathbb{R}.$$

- Inverse sec function:
$$y = \sec x \qquad \Leftrightarrow \qquad x = \sec^{-1} y \qquad \text{for } x \in [0, \pi/2) \cup [\pi, 3\pi/2), \; y \in (-\infty, -1] \cup [1, \infty).$$

- Inverse cosec function:
$$y = \csc x \qquad \Leftrightarrow \qquad x = \csc^{-1} y \qquad \text{for } x \in (0, \pi/2] \cup (\pi, 3\pi/2] \; y \in (-\infty, -1] \cup [1, \infty).$$

Concept 4.9.2 Derivatives of inverse trigonometric functions.
Note that inverse trig functions are also denoted by $\arcsin(x)$, $\arccos(x)$, etc., which is often preferred. And, very importantly, remark that

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}, \qquad \frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}},$$

$$\sin^{-1}(x) \neq \frac{1}{\sin(x)} \; !$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2},$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}.$$

4.9.3 Further readings

Section 2.12 in CLP1

Chapter 5

Integration

5.1 Antiderivatives and indefinite integrals

We are now masters of differentiation. Given a function, we know how to calculate its derivative. Great! But what about the inverse process? Suppose that you are given a function $f(x)$: what other functions $F(x)$ are such that their derivatives $F'(x)$ are equal to the original function $f(x)$? From the point of view of kinematics: suppose that you know the velocity function of an object, can you determine its position function? From the point of view of geometry: suppose that you know the slope of the tangent lines to the graph of a function at all points, can you determine what the original function is?[1]

This is the idea behind integration. In this section we take the first steps. We define the concept of "antiderivatives": an antiderivative of a function f is another function F such that its derivative is equal to the original function f. We also introduce the idea of "indefinite integral", denoted by $\int f(x)\ dx$, which represents the most general antiderivative of a function f.

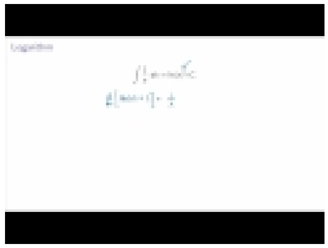
Objectives

You should be able to:

- Explain the meaning of an antiderivative and an indefinite integral.
- Use correct notation for antiderivatives and indefinite integrals.
- Recall the antiderivatives of elementary functions.
- Evaluate indefinite integrals of simple functions.

[1]The answer to these last two questions is: almost. As we will see, you would also need one more piece information to answer unambiguously these questions: either the position of the object at a certain time, or the coordinates of a point that the graph of the function passes through. This would unambiguously fix the appropriate antiderivative, or, in other words, fix the "constant of integration", as we will see.

5.1.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=5-xjQZdYmus`

5.1.2 Key concepts

Concept 5.1.1 Antiderivatives. A function F is called an **antiderivative of f on an interval** I if $F'(x) = f(x)$ for all $x \in I$.

If F is an antiderivative of f on I, then the most general antiderivative of f on I is

$$F(x) + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Concept 5.1.2 Indefinite integrals. We use the notation

$$\int f(x)\ dx = F(x) + C$$

to denote the most general antiderivative of f; this expression is called the **indefinite integral of f**.

The notation $\int f(x)\ dx$ means "finding the general antiderivative of $f(x)$," just as $\frac{d}{dx}f(x)$ means "finding the derivative of $f(x)$."

Concept 5.1.3 Table of indefinite integrals. First, we note the following two important properties of indefinite integrals:

$$\begin{aligned} \int c\ f(x)\ dx =& c\int f(x)\ dx \qquad \text{for } c \text{ a constant,} \\ \int \left(f(x) + g(x)\right)\ dx =& \int f(x)\ dx + \int g(x)\ dx. \end{aligned}$$

Next, we list below a few well known indefinite integrals. These formulae can be verified by differentiating the right-hand-side and obtaining the integrand on the left-hand-side (the thing inside the indefinite integral, without the dx).

$$\begin{aligned} \int x^n\ dx =& \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1) \\ \int \frac{1}{x}\ dx =& \ln|x| + C \\ \int e^x\ dx =& e^x + C \\ \int a^x\ dx =& \frac{a^x}{\ln(a)} + C \\ \int \sin x\ dx =& -\cos x + C \\ \int \cos x\ dx =& \sin x + C \end{aligned}$$

$$\int \sec^2 x \; dx = \tan x + C$$
$$\int \csc^2 x \; dx = -\cot x + C$$
$$\int \sec x \tan x \; dx = \sec x + C$$
$$\int \csc x \cot x \; dx = -\csc x + C$$
$$\int \frac{1}{x^2+1} \; dx = \tan^{-1} x + C$$
$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1} x + C$$
$$\int \frac{1}{x\sqrt{x^2-1}} \; dx = \sec^{-1} x + C$$

5.1.3 Further readings

Section 4.1 in CLP1

5.2 Area, displacement and Riemann sums

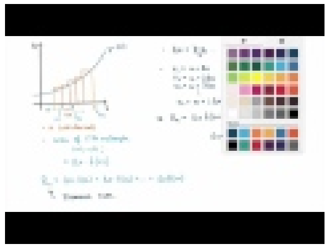
We now go back to the beginning. In our preview of calculus (Section 2.1), we saw that we can estimate the area under a curve by replacing it with a finite number of rectangles of appropriate heights and widths. In this section we study this approach more rigourously; it gives rise to the concept of "Riemann sums". We introduce the "summation notation", study some of its properties, and use it to calculate a few simple Riemann sums. We also explore how taking the limit of a Riemann sum where the number of rectangles becomes infinite (in which case their widths go to zero) gives rise to a precise definition of the area under a curve.

Objectives

You should be able to:

- Describe and illustrate how to approximate the area under a curve using approximating rectangles and a Riemann sum.
- Construct a Riemann sum to approximate the area under the curve of a given function over a given interval $[a, b]$ using n subintervals, with either left endpoints, right endpoints, or mid endpoints.
- Calculate the value of a Riemann sum for a given function over a given interval for a given value of n.
- Describe the limit process that arises in the calculation of the precise area under a curve using Riemann sums.
- Calculate the area under a curve using limits of Riemann sums.
- Evaluate simple finite sums using the summation notation.

5.2.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=FffzDjY8Hiw`

5.2.2 Key concepts

Concept 5.2.1 Summation notation. For $\{a_i\}$ a set of numbers indexed by the integers, and for integers $k \leq n$ we have

$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + \cdots + a_{n-1} + a_n.$$

Concept 5.2.2 Properties of sums.

$$\sum_{i=k}^{n} ca_i = c\sum_{i=k}^{n} a_i, \qquad \sum_{i=k}^{n}(a_i \pm b_i) = \sum_{i=k}^{n} a_i \pm \sum_{i=k}^{n} b_i,$$

and for any integer k with $a < k < b$,

$$\sum_{i=a}^{b} c_i = \sum_{i=a}^{k} c_i + \sum_{i=k+1}^{b} c_i.$$

Concept 5.2.3 Four useful finite sums.

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Concept 5.2.4 Riemann sums. For $f(x) \geq 0$ on the interval $[a, b]$, to calculate the approximate area bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ using n intervals of equal width and the right endpoints of each interval we define the Riemann sum:

$$R_n = \sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x), \qquad \text{where} \quad \Delta x = (b - a)/n.$$

When n is increased the number of rectangles used is increased and the approximation is improved.

Concept 5.2.5 Right endpoints, left endpoints, and mid endpoints. This is the Riemann sum for “right endpoints”, meaning that the rectangles approximating the area under the curve have heights $f(x_i)$ with x_i corresponding to the right endpoints of the rectangles. We can also define similarly Riemann sums L_n for “left endpoints”, and M_n for “mid endpoints”.

Concept 5.2.6 Limit of infinite number of rectangles of zero width. If we let $n \to \infty$, in whic case $\Delta x \to 0$, then the limit of all the Riemann sums R_n, L_n, and M_n (right endpoint, left endpoint, and mid endpoint) are all equal:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n.$$

This limit defines the ***exact area under the curve*** A .

Concept 5.2.7 Interpretation in kinematics. If $f(t) \geq 0$ is a velocity function then the Riemann sum R_n (or L_n or M_n) can be used to approximate the distance traveled during the interval of time from $t = a$ to $t = b$, and $\lim_{n\to\infty} R_n$ becomes the exact distance traveled.

5.2.3 Further readings

Section 1.1 in CLP2

5.3 Definite integrals

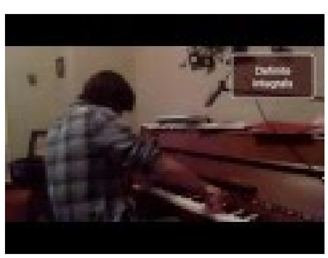
We now study in more detail the limit of Riemann sums as the number of rectangles go to infinity. The result is known as a "definite integral". We introduce appropriate notation for definite integrals, and study their properties. In particular, we clarify the relation between the definite integral and the area under the curve in the case where the function $f(x)$ is not necessarily positive.

Objectives

You should be able to:

- Use correct notation for the definite integral and the limit process to represent the area under a curve.
- Calculate the value of a definite integral using an appropriate limit of a Riemann sum.
- Calculate the value of a definite integral using the interpretation of the definite integral as a net area and the properties of the definite integral.
- Generate and/or explain properties of the definite integral based on the interpretation of the definite integral as a net area (example: the integral of an odd function over an interval that is symmetric about the origin is zero).

5.3.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=rJ3JX5V8Z24`

5.3.2 Key concepts

Concept 5.3.1 Definite integrals. Let $f(x)$ be a function defined for $x \in [a, b]$. We divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, $x_1 = a + \Delta x, \ldots, x_n = b$ be the right endpoints of these intervals. The Riemann sum R_n is defined by

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The **definite integral of f from a to b**, denoted by $\int_a^b f(x)\, dx$, is the $n \to \infty$ limit of R_n:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \qquad \text{with } \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x,$$

provided that the limit exists. If it exists, we say that f is **integrable** on $[a, b]$.

In the definition above we used the right-point rule to write down the Riemann sum R_n. But in fact we can use any point $x_i^* \in [x_{i-1}, x_i]$ in the subintervals to define the Riemann sum:

$$S_n = \sum_{i=1}^{n} \Delta x f(x_i^*).$$

The definite integral is still obtained as the $n \to \infty$ limit of S_n, and it is equal to the definition above, regardless of the choice of x_i^*.

In the notation $\int_a^b f(x)\, dx$, $f(x)$ is called the **integrand**, and a and b are called the **limits of integration**: a is the **lower limit** while b is the **upper limit**.

Note: the definite integral $\int_a^b f(x)\, dx$ is a number; it is not a function of x.

Concept 5.3.2 Integrable functions. Many functions are integrable. More precisely, if f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities on $[a, b]$, then it is integrable on $[a, b]$.

Concept 5.3.3 Definite integrals and areas. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\, dx$ calculates the area bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$.

If $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)\, dx$ calculates minus the area bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$.

In general, if $f(x)$ is partly positive and partly negative over $[a, b]$, then $\int_a^b f(x)\, dx$ calculates the **net area**, which is the area above the x-axis minus the area below the x-axis.

Accordingly, the **true area** (as opposed to the net area) between $y = f(x)$, $y = 0$, $x = a$ and $x = b$ is given by

$$A = \int_a^b |f(x)|\, dx.$$

Concept 5.3.4 Properties of definite integrals. Many properties of definite integrals can be proved from the geometric interpretation:

1. $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx,$

2. $\displaystyle\int_a^a f(x)\, dx = 0,$

3. $\int_a^b c\, f(x)\ dx = c \int_a^b f(x)\ dx$,

4. $\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$,

5. $\int_a^b dx = b - a$,

6. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$,

7. If $f(x)$ is even, then $\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx$,

8. If $f(x)$ is odd, then $\int_{-a}^a f(x)\ dx = 0$,

9. If $f(x) \geq 0$ for $x \in [a, b]$, then $\int_a^b f(x)\ dx \geq 0$,

10. If $f(x) \geq g(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$,

11. If $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a),$$

12.

$$\left| \int_a^b f(x)\ dx \right| \leq \int_a^b |f(x)|\ dx.$$

5.3.3 Further readings

Section 1.1 in CLP2Section 1.2 in CLP2

5.4 The Fundamental Theorem of Calculus

We are now getting into the truly amazing core of calculus. We mentioned a number of times that “differentiation and integration are inverse processes”. But what does this mean precisely? This is the meat of the Fundamental Theorem of Calculus.

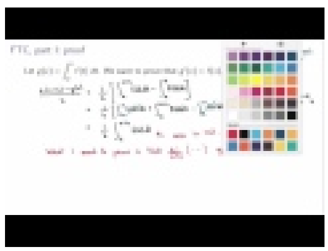
In fact, in previous sections, we used pretty much the same symbol, $\int$, to first denote the indefinite integral of a function (which is its most general antiderivative - this was the symbol without limits of integration), and second the definite integral of a function (which is the limit of a Riemann sum - this was the symbol with limits of integration). A priori, those are two very different concepts. Why do we use the same symbol for both? This is because they are very intimately related: it turns out that we can evaluate definite integrals in terms of arbitrary antiderivatives. Again, this is encapsulated into the all-important Fundamental Theorem of Calculus. So let’s dive right into it!

Objectives

You should be able to:

- State the Fundamental Theorem of Calculus (FTC).
- Sketch the main lines of the proof of the FTC.
- Explain in which sense the FTC is saying that differentiation and integration are inverse processes.
- Use the FTC Part 1, in conjunction with the chain rule and properties of definite integrals, to evaluate the derivatives of functions presented as integrals.
- Use the FTC Part 2 to evaluate integrals in terms of antiderivatives.

5.4.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=C7Z_wET0blY`

5.4.2 Key concepts

Concept 5.4.1 The Fundamental Theorem of Calculus (FTC). Let $f(x)$ be a continuous function on $[a, b]$. Then:

1. If $g(x) = \displaystyle\int_a^x f(t)\ dt$, for $a \le x \le b$, then $g'(x) = f(x)$;
2. $\displaystyle\int_a^b f(x)\ dx = F(b) - F(a)$, where F is an arbitrary antiderivative of f.

The FTC gives a precise meaning to the statement that integration and differentiation are inverse processes.

Concept 5.4.2 Some uses of the FTC.

1. The FTC part 1 can be used to evaluate derivatives of functions that are given in integral form. Note that in part 1 above, $g(x)$ is a function of x, not of the dummy variable t that is integrated over.
2. The FTC part 2 can be used to evaluate definite integrals, by first finding an antiderivative of the integrand.

5.4.3 Further readings

Section 1.3 in CLP2

5.5 Substitution

Now the we understand a bit more about integration, we can start addressing the problem of evaluating integrals. Using the Fundamental Theorem of Calculus, we know that to evaluate definite integrals, we need to find an antiderivative of the integrand. But finding antiderivatives is not always obvious. In fact, it is a highly non-trivial problem in general; finding an antiderivative of a function is much more difficult than finding the derivative of a function, as the former is not algorithmic, while the latter is. To find the derivative of a function, you just need to apply repeatedly differentiation rules; but there is no precise recipe that I can give you to find antiderivatives of functions in general.

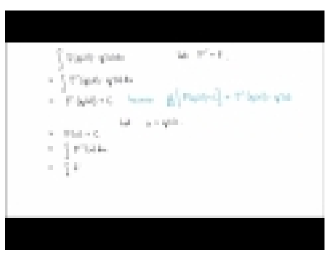
However, all is not lost. There is a number of techniques that we can develop to help find antiderivatives of complicated functions. In this section we explore our first technique of integration, known as "substitution". Other techniques of integration will be covered in MATH 146. Note that substitution is by far the most useful technique of integration: one that you will use over and over again in your mathematical life.

Objectives

You should be able to:

- Evaluate indefinite integrals using substitution.
- Evaluate definite integrals using substitution and the Fundamental Theorem of Calculus.
- Explain in what sense substitution undoes the chain rule.

5.5.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=vZbBbNyYfbg`

5.5.2 Key concepts

Concept 5.5.1 The substitution rule. If $u = g(x)$ is a differentiable function and $f(x)$ is continuous over the range of $g(x)$, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

In other words, the substitution $u = g(x)$, with $du = g'(x)dx$, "undoes" the chain rule

In practice, what this means is that you can do a substitution $u = g(x)$, $du = g'(x)dx$ inside an integral. This will be useful if you can then rewrite the integrand as a function of u that is easier to integrate than the original integrand as a function of x.

Concept 5.5.2 Substitution for definite integrals. Substitution also works for definite integrals, but one has to be careful with the limits of integration. There are two methods to evaluate definite integrals using substitution. The first one is often the preferred method.

1. The idea is to transform the limits of integration from x-values to u-values as you perform the substitution:

$$\int_a^b f((g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du, \qquad \text{with } u = g(x), du = g'(x)dx.$$

   Then you can evaluate the resulting definite integral in u directly using the Fundamental Theorem of Calculus (i.e. by finding an antiderivative of $f(u)$).

2. The second method is to first find an antiderivative of the integrand using substitution, rewrite it in terms of the original variable x, and then evaluate at the limits of integration $x = a$ and $x = b$ using the Fundamental Theorem of Calculus. This works as well, but you have to be careful with notation and make sure that you rewrite everything in terms of the x-variable before you evaluate at the limits of integration.

5.5.3 Further readings

Section 1.4 in CLP2

5.6 Areas between curves

We introduced Riemann sums in Section 2.1 and Section 5.2 as approximations of the area under the graph of a positive function. Then, we argued in Section 5.3 that by taking the limit where the number of rectangles becomes infinite, we obtain a precise calculation of the area. This is the fundamental idea behind Riemann sums and definite integrals.

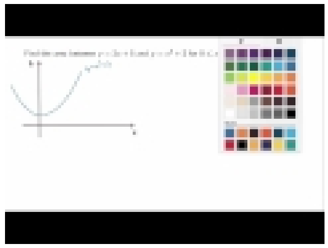
In this section we study in more detail the intimate connection between definite integrals and areas between curves. In particular, we drop the assumption that the curve is given by the graph of a positive function. This is our first application of integration, in this case to geometry. We will see many more applications of integration in MATH 146.

Objectives

You should be able to:

- Describe why the area bounded by the graphs of two functions can be written as a definite integral.
- Write down and evaluate the definite integral representing the area of a given planar region.
- Differentiate between geometrical situations where integration in y is more appropriate than integration in x to calculate the area of a given planar region.
- Determine when it is necessary to split a given planar region into subregions and use more than one integral to evaluate the area.

5.6.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=KGJ6aM87mCI`

5.6.2 Key concepts

Concept 5.6.1 Definite integrals and areas between curves. The area A of the region S between the curves $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$ and $x = b$, with $a \leq b$, is given by the definite integral

$$A = \int_a^b |f(x) - g(x)| \, dx.$$

This expression can be obtained by slicing the region into n rectangles of equal width Δx. The Riemann sum then gives an approximation of the area S, and the limit $n \to \infty$ calculates A. To recover the integral expression directly, draw a typical rectangle, with width dx and height $|f(x) - g(x)|$: the area is given by integrating over typical rectangles from $x = a$ to $x = b$.

If $f(x) \geq g(x)$ over the interval $[a, b]$, the integral simplifies to

$$A = \int_a^b (f(x) - g(x)) dx.$$

In the general case, where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, to evaluate the integral above we split the region S into smaller subregions $S_1, S_2, \ldots, S_n$ where either $f(x) \geq g(x)$ or $g(x) \geq f(x)$ over a given subregion. Then we can use the fact that

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x), \\ g(x) - f(x) & \text{if } g(x) \geq f(x), \end{cases}$$

to evaluate the area A_i of each subregion S_i, and add them up to get the area $A = A_1 + A_2 + \ldots + A_n$.

Concept 5.6.2 Vertical vs horizontal slicing. Sometimes it is better to calculate the area of a region by slicing it with horizontal rectangles. If the region S is bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, with $c \leq d$, then a typical horizontal rectangle will have height dy and width $|f(y) - g(y)|$, and the area will be given by integrating from $y = c$ to $y = d$:

$$A = \int_c^d |f(y) - g(y)| \, dy.$$

If $f(y) \geq g(y)$ over the interval $y \in [c, d]$ (that is, the curve $x = f(y)$ is the right boundary of the region, while the curve $x = g(y)$ is the left boundary), then you can drop the absolute value in the expression above.

5.6.3 Further readings

Section 1.5 in CLP2

Chapter 6

Functions and curves

6.1 The Intermediate Value Theorem

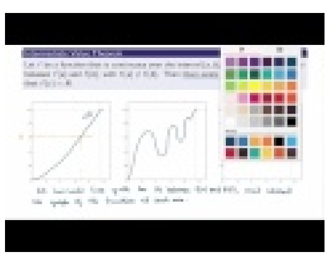
After our brief exploration of integrals, we now go back to our study of functions and their properties. In Section 3.5 we introduced the concept of "continuous functions". In this section we explore a fundamental property that is satisfied by all continuous functions, known as the "Intermediate Value Theorem" (IVT). While the statement of theorem may seem rather obvious, its proof is quite involved, and its applications are far-reaching, in fact even surprising, as we will see! In particular, one application of the IVT is to locate the roots of complicated functions: repeated applications of the IVT gives rises to a numerical root finding method, known as the "bisection method".

Objectives

You should be able to:

- Explain and illustrate the Intermediate Value Theorem.
- Apply the Intermediate Value Theorem to locate the roots of a function.
- Solve simple problems using the Intermediate Value Theorem.

6.1.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=EqcBS56nYbg`

6.1.2 Key concepts

Concept 6.1.1 Intermediate Value Theorem (IVT). Let f be a continuous function over the interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, with $f(a) \neq f(b)$. Then the **Intermediate Value Theorem** states that there must exist a $c \in (a, b)$ such that $f(c) = N$.

Equivalently, for any N between $f(a)$ and $f(b)$, the horizontal line $y = N$ must intersect the graph of f at least once.

Two things to note:

- c may not be unique;
- If f is not continuous, then the statement of the Intermediate Value Theorem may not hold.

Concept 6.1.2 Using the IVT to locate roots of functions. To locate the roots of a continuous function f using the Intermediate Value Theorem, pick two values of x, say $x = a$ and $x = b$, such that $f(a)$ is negative and $f(b)$ is positive. Then the Intermediate Value Theorem implies that f must have at least one root between $x = a$ and $x = b$.

You can repeat the process using the midpoint of the previous interval to get a better numerical approximation for the location of the root: this is known as the **bisection method**.

6.1.3 Further readings

Section 1.6.3 in CLP1

6.2 The Mean Value Theorem

Continuous functions satisfy the Intermediate Value Theorem; well, differentiable functions also satisfy their own, nice, theorem, known as the "Mean Value Theorem" (MVT). This is what we explore in this section. While it may seem daunting at first, the statement of the MVT is in the end fairly obvious. However, it has fundamental consequences. For instance, it follows from the MVT that any function whose derivative vanishes over some interval must be constant over that interval. As a consequence, it follows that two antiderivatives of a given function can only differ by the addition of a constant: a statement that we have used repeatedly! Using the MVT, we can now understand why this statement must be true.

Objectives

You should be able to:

- Explain and illustrate the Mean Value Theorem.
- Relate the Mean Value Theorem to the notions of average and instantaneous velocities.
- Solve simple problems using the Mean Value Theorem.

6.2.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=oobkaT09Vmo`

6.2.2 Key concepts

Concept 6.2.1 Mean Value Theorem (MVT). Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then the **Mean Value Theorem** states that there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, the Mean Value Theorem is saying that there must be a $c \in (a, b)$ such that the tangent line to $y = f(x)$ at $x = c$ is parallel to the secant line between $(a, f(a))$ and $(b, f(b))$.

It can also be understood as saying that there must be a $c \in (a, b)$ at which the instantaneous rate of change of f is equal to its average rate of change between a and b.

Note that as for the Intermediate Value Theorem, c need not be unique; there may be more than one c satisfying the statement of the Mean Value Theorem.

Concept 6.2.2 Rolle's Theorem. In the special case where $f(a) = f(b)$ (the secant line is horizontal), then the statement becomes that there must be a $c \in (a, b)$ such that $f'(c) = 0$ (the tangent line is horizontal), which is called **Rolle's Theorem**.

Concept 6.2.3 Important consequences of the Mean Value Theorem. The following statements are consequences of the Mean Value Theorem:

- If $f'(x) = 0$ for all x in some interval (a, b), then f must be constant on (a, b).
- If $f'(x) = g'(x)$ for all x in some interval (a, b), then $f - g$ must be constant on (a, b). That is, $f(x) = g(x) + C$ for some constant C.

6.2.3 Further readings

Section 2.13 in CLP1

6.3 Maxima and minima

In this section we study how to find maxima and minima of functions. We distinguish between local and absolute extrema of a function. We study the Extreme Value Theorem, which provides a simple approach to finding the absolute max and min of a function on a closed interval. Overall, the content of this section is very useful to sketch the graph of complicated functions (Section 6.4), but also to solve optimization problems, which are very common in science, economics, etc. (Section 7.2)

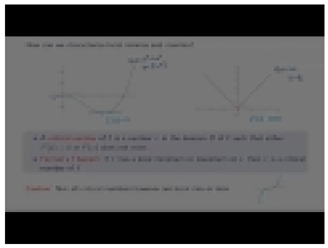
Objectives

You should be able to:

- Explain and illustrate the definition of local and absolute extrema.
- Explain and illustrate the definition of critical numbers of a function.
- Relate critical numbers of a function to its local extrema.

- Find the critical numbers of a function.
- Explain and illustrate geometrically the Extreme Value Theorem.
- Find the absolute minimum and maximum of a function on a closed interval.

6.3.1 Instructional video



YouTube: https://www.youtube.com/watch?v=wEUjRKOeTyo

6.3.2 Key concepts

Concept 6.3.1 Min and max of a function. Let $c \in D$ where D is the domain of f. Then $f(c)$ is:

- The **absolute maximum** of f on D if $f(c) \geq f(x)$ for all $x \in D$;
- The **absolute minimum** of f on D if $f(c) \leq f(x)$ for all $x \in D$;
- The **local maximum** of f on D if $f(c) \geq f(x)$ for all x near c;
- The **local minimum** of f on D if $f(c) \leq f(x)$ for all x near c.

Near c means "for all x in some open interval containing c."

In general, absolute minima and maxima can occur either at local minima or maxima, or, if D is a closed interval, at the endpoints of the interval.

Concept 6.3.2 Extrema and critical numbers. A **critical number** of f is a number c in the domain D of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

If f has a local minimum or maximum at c, then c is a critical number of f (Fermat's Theorem). However, the converse is not true: not all critical numbers are local minima or maxima.

Concept 6.3.3 Extreme Value Theorem. If f is continuous on $[a, b]$, then f must attain an absolute maximum $f(c)$ and an absolute minimum $f(d)$ at some numbers $c, d \in [a, b]$.

Concept 6.3.4 How to find the absolute extrema of a continuous function f on a closed interval $[a, b]$.

1. Find the critical numbers of f in (a, b), and evaluate f at the critical numbers;
2. Evaluate f at the endpoints a and b;
3. Compare the values. The largest of these values is the absolute maximum, while the smallest is the absolute minimum.

6.3.3 Further readings

Section 3.5 in CLP1

6.4 Curve sketching

We now have all the information needed to sketch the graph of complicated functions by hand. In this section we provide a ten-step method for sketching the graph of a function f, using the information provide by f, its derivative f', and its second derivative f''.

Objectives

You should be able to:

- Find the domain of a function.
- Find the intercepts of a function.
- Determine the intervals where a function is positive or negative.
- Determine whether a function is even, odd, or periodic.
- Find the vertical, horizontal and slant asymptotes of a function.
- Find the intervals where a function is increasing and decreasing by studying the first derivative of the function.
- Find the local minima and maxima of a function (determine whether a function has a local min or max at a given critical number using the First Derivative Test and Second Derivative Test).
- Find the intervals where a function is concave up and concave down by studying the second derivative of the function.
- Find the inflection points of a function.
- Sketch a detailed graph of a given function using information as determined in the 9 steps above.

6.4.1 Instructional videos

YouTube: `https://www.youtube.com/watch?v=Vyhui3AV8FA`

YouTube: `https://www.youtube.com/watch?v=H1ENyjqn3Tw`

6.4.2 Key concepts

Concept 6.4.1 Increasing/Decreasing Test. Given a function f, one can find the intervals where the function is increasing and decreasing using information from f':

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Concept 6.4.2 First Derivative Test. The first derivative test is useful to determine whether a critical number of a function f is a local min or max, using information from f'.

Let c be a critical number of a continuous function f.

- If f' changes from positive to negative at c, then f has a local max at c.
- If f' changes from negative to positive at c, then f has a local min at c.
- If f' does not change sign at c, then f has no local extremum at c.

Concept 6.4.3 Concavity Test. Given a function f, one can find the intervals where the function is concave up or down using information from f'':

- If $f''(x) > 0$ on an interval, then f is concave up on that interval (happy).
- If $f''(x) < 0$ on an interval, then f is concave down on that interval (unhappy).

A point where the graph of a function is continuous and where the concavity changes is called an **inflection point** of f.

Concept 6.4.4 Second Derivative Test. The second derivative test is useful to determin whether a critical number of a function f is a local min or max, using information from f''

Suppose f'' is continuous near c:

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

Note that if $f'(c) = 0$ and $f''(c) = 0$, then the test is not conclusive: f may have a local max, a local min, or neither.

Concept 6.4.5 Vertical asymptotes. The vertical line $x = a$ is a **vertical asymptote** of $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

Concept 6.4.6 Horizontal asymptotes. The horizontal line $y = L$ is a

horizontal asymptote of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L, \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Concept 6.4.7 Slant asymptotes. Slant asymptotes occur when a function $y = f(x)$ approaches a line (that is not horizontal) as x goes to $\pm\infty$.

The line $y = mx + b$, $m \neq 0$, is a **slant asymptote** of $y = f(x)$ if either

$$\lim_{x\to\infty} (f(x) - (mx + b)) = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} (f(x) - (mx + b)) = 0.$$

This says that the vertical distance between the graph of $y = f(x)$ and the line $y = mx + b$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

Slant asymptotes are commonly found when $f(x)$ is a rational function, and the degree of the numerator is one more than the degree of the denominator. Then, it can be found by performing long division for the rational function.

Concept 6.4.8 Summary of curve sketching. Ten step method:

Information from $f(x)$:

1. **Domain:** find the domain of f.
2. **Intercepts:** Find the y-intercepts $(0, f(0))$ and x-intercepts (the points where $f(x) = 0$).
3. **Positivity:** Find where f is positive and negative.
4. **Symmetry:** Is f even or odd? Is f periodic?
5. **Asymptotes:** Find the vertical, horizontal and slant asymptotes of f if any exist.

Information from $f'(x)$*:*

6. **Intervals of increase and decrease:** Find $f'(x)$ and determine when $f'(x) > 0$ (f is increasing) and when $f'(x) < 0$ (f is decreasing).
7. **Local max and min points:** Find the critical points of f, if any, and identify the local max and min.

Information from $f''(x)$*:*

8. **Concavity:** Find $f''(x)$, determine when $f''(x) > 0$ (f is concave up) and when $f''(x) < 0$ (f is concave down).
9. **Inflection points:** Find the inflection points of f, if any.

And finally...

10. **Sketch the graph:** Plot key points (intercepts, critical points, local max and min, points of inflection), and sketch the curve together with its asymptotes.

6.4.3 Further readings

Section 3.6 in CLP1

Chapter 7

Applications of differentiation

7.1 Related rates

We now study a few applications of differentiation. This is fun stuff! Our first application concerns problems involving relationships between quantities that are changing in time. Such problems are called “related rate” problems, and can usually be solved via differentiation.

Objectives

You should be able to:

- Introduce notation to express a written problem involving rates of change in terms of derivatives.
- Solve various problems from physics, chemistry, biology, etc., involving relationships between changing quantities by applying differentiation techniques (related rates problems).

7.1.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=goTaQeHmzoI`

7.1.2 Key concepts

Concept 7.1.1 Strategy to solve related problems.

1. Read the problem carefully.
2. Draw a picture if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the desired rate of change in terms of

derivatives.

5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.

6. Differentiate both sides of the equation with respect to t.

7. Substitute the given information into the new equation and solve for the desired rate of change.

7.1.3 Further readings

Section 3.2 in CLP1

7.2 Optimization

Our next application concerns “optimization problems”. Suppose that you are interested in maximizing (or minimizing) a certain quantity, which may depend on a number of factors. How can you determine how these factors should be adjusted so that the quantity of interest is maximized (or minimized)? It turns out that differentiation is the tool of choice for answering questions of this type.

Objectives

You should be able to:

- Introduce notation to transform an optimization problem into a calculus question about finding the extremum of a function.

- Solve various optimization problems from physics, chemistry, economics, population dynamics, etc. by applying calculus techniques to find the extremum of a function.

7.2.1 Instructional video

YouTube: `https://www.youtube.com/watch?v=fzflFNwp5GQ`

7.2.2 Key concepts

Concept 7.2.1 Strategy to solve optimization problems.

1. Read the problem carefully.

2. Draw a picture if possible.

3. Introduce notation. Assign a symbol (say Q) to the quantity that is to be maximized or minimized, and symbols to the other unknown quantities relevant to the problem.

4. Write an equation for Q in terms of the other unknowns of the problem. If Q is expressed in terms of more than one variables, use the information of the problem to find relationships between these variables, and eliminate all but one the variables in the expression for Q. Then Q will be expressed as a function of a single variable, say $Q = f(x)$.

5. Find the absolute maximum or minimum value of $f(x)$ over the range of x allowed by the problem. If this is a closed interval in x, you can use the closed interval method, otherwise, you need to find the local min or max and justify why a given local min or a max is the absolute max or min of $f(x)$.

7.2.3 Further readings

Section 3.5 in CLP1

7.3 Linear approximation

When we introduced the concept of derivative in Chapter 2 and Section 4.1, we studied how the derivative can be understood geometrically as calculating the slope of the tangent line to the graph of a function at a point. Moreover, we mentioned that if we zoom in on the graph of the function near this point, the tangent line becomes closer and closer to the graph of the function. In other words, it provides a fairly good approximation of the function near that point. This is the idea of "linear approximations".

In fact, calculus could be understood as a theory of approximations: the derivative is a machinery that allows us to construct polynomial approximations of functions. The linear approximation, which corresponds to the tangent line, is the degree one polynomial approximation of a function at a point.

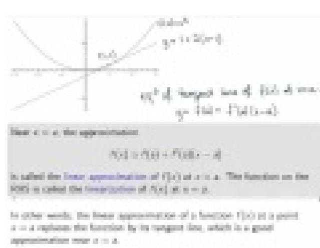
In this section we study the idea of linear approximations in more detail, and explore how it can be applied in various problems in the physical sciences.

Objectives

You should be able to:

- Find the linear approximation of a function at a given point.
- Illustrate the relation between the function and its linear approximation.
- Use the linear approximation of a function at a point to approximate the value of the function near this point.
- Deduce whether a linear approximation over- or under-estimates the exact value of a function from information about concavity of the function.

7.3.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=sgbPy_tkfwA`

7.3.2 Key concepts

Concept 7.3.1 Linear approximation and linearization. Recall that the tangent line of a differentiable function $f(x)$ at point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a).$$

For x values very close to a the graph of $f(x)$ is very close to the graph of the tangent line. Then, we can use the tangent line to obtain the approximation

$$f(x) \approx f(a) + f'(a)(x - a);$$

this is called the **linear approximation** of $f(x)$ at a.

The function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization of f at a**.

7.3.3 Further readings

Section 3.4 in CLP1

7.4 Taylor polynomials

In the previous section we studied the linear approximation of a function at a point. But why stop at linear polynomials? Wouldn't we be able to achieve better approximations if we looked for higher degree polynomials?

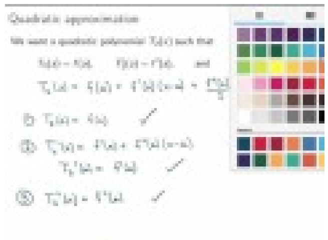
The answer is yes, and the result is what is called the "Taylor polynomials" of a function at a point. The degree d Taylor polynomial of a function at a point approximates that function near this point by a degree d polynomial. It turns out that the Taylor polynomials of a function at a point can be fully calculated using differentiation. I told you that calculus was a theory of approximations!

Objectives

You should be able to:

- Calculate the Taylor polynomials of a function, and compute its corresponding higher degree approximation at a given point.
- Illustrate the relation between the function and its higher degree approximations.
- Use the higher degree approximations of a function at a point to approximate the value of the function near this point.

7.4.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=anEgFWVe1vQ`

7.4.2 Key concepts

Concept 7.4.1 Taylor polynomials. The idea of Taylor polynomials is to approximate a function $f(x)$ at a point $x = a$ by higher degree polynomials. The polynomial

$$T_d(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(d)}(a)}{d!}(x-a)^d,$$

where $k! = 1 \cdot 2 \cdot 3 \ldots k$, is called the **degree d Taylor polynomial** of $f(x)$ at $x = a$.

It satisfies the properties:

$$T_d(a) = f(a) \qquad \text{and} \qquad T_d^{(k)}(a) = f^{(k)}(a)$$

for all $k = 1, 2, \ldots, d$.

Concept 7.4.2 The degree d polynomial approximation of a function at a point. The degree d approximation of $f(x)$ at $x = a$ is

$$f(x) \approx T_d(x),$$

where $T_d(x)$ is the degree d Taylor polynomial of $f(x)$ at $x = a$. When $d = 1$ we get the linear approximation of f at $x = a$,

$$f(x) \approx f(a) + f'(a)(x-a).$$

When $d = 2$ we get the quadratic approximation of f at $x = a$,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Concept 7.4.3 The Taylor series. In fact, we could try to send $d \to \infty$; that would give an "approximation" of $f(x)$ at $x = a$ by an "infinite degree polynomial". This can be made rigorous using sequence and series; namely, taking $d \to \infty$ means taking the limit of the sequence of Taylor polynomials $T_d(x)$, and the result is the **Taylor series of $f(x)$ at $x = a$**. Reversing the process, one can think of the Taylor polynomials above as truncations (so-called "partial sums") of the Taylor series of $f(x)$ at $x = a$.

7.4.3 Further readings

Section 3.4 in CLP1

7.5 Newton's method

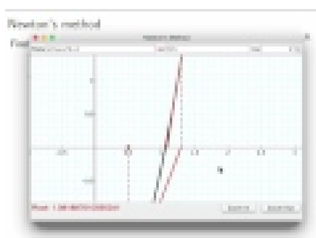
Our final application of differentiation is a very cool application of linear approximations. We study how repeated applications of linear approximations give rise to a very efficient numerical algorithm for finding roots of functions, known as "Newton's method". We have already seen in Section 6.1 another method for finding roots of a function, namely the bisection method, which resulted from repeated applications of the Intermediate Value Theorem. It turns out that Newton's method is a much more efficient algorithm for finding roots of a function (however, Newton's method may sometime fail for some choices of initial conditions). In fact, one can use Newton's method to calculate square roots, such as $\sqrt{42}$, to fairly good accuracy, in your head!

Objectives

You should be able to:

- Explain and illustrate Newton's method.
- Apply Newton's method to find approximations of all roots of a given function correct to a given number of decimal places.
- Explain and illustrate why Newton's method may fail for certain choices of initial conditions.

7.5.1 Instructional video



YouTube: `https://www.youtube.com/watch?v=LZrATWBLLdk`

7.5.2 Key concepts

Concept 7.5.1 Newton's method. To find a root r of a differentiable function $f(x)$ near an x-value of x_1, we apply **Newton's Method**. The idea is to replace the function by its linearization at x_1, and then find the x-intercept x_2 of the linearization, which is a first approximation of the root of $f(x)$. We then repeat the process at x_2, and so on, until we reach a desired numerical precision for the root of $f(x)$. More precisely:

1. First we calculate x_2, the x-intercept of the tangent line of f at $(x_1, f(x_1))$ (the linearization of f at x_1). We get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

2. Then we find x_3, the x-intercept of the tangent line of f at $(x_2, f(x_2))$, and get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

3. We can repeat this process as many times as desired. At each step, the x-intercept of the tangent line of f at $(x_n, f(x_n))$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

4. If we want an answer valid for k decimal places, we stop the process when two successive values x_n and x_{n+1} have the exact same first k decimals. We conclude that x_{n+1} is then a root of $f(x)$, up to a precision of $k >$ decimal places.

Note that Newton's method may sometimes converge very slowly. In fact, it can also fail, for a number of reasons, for instance if the starting point x_1 is not chosen appropriately.